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Integer Programming Approaches To Subspace Clustering With Missing Data

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Abstract

In the *Subspace Clustering with Missing Data* (SCMD) problem, we are given a collection of n data points, arranged into columns of a matrix $X \in \mathbb{R}^{d\times n}$, and each of the data points is observed only on a subset of its coordinates. The data points are assumed to be concentrated near a union of low-dimensional subspaces. The goal of SCMD is to cluster the vectors of the data matrix X as per their subspace membership. State-of-the-art algorithms for SCMD can perform poorly on instances with a large amount of missing data or if the data matrix X is nearly full-rank. We propose a novel integer programming-based method for SCMD. The approach is based on dynamically determining a set of candidate subspaces and optimally assigning points to selected subspaces. The problem structure is identical to the classical facility-location problem, with subspaces playing the role of facilities and data points that of customers. We propose a column-generation approach for identifying candidate subspaces combined with a Benders decomposition approach for solving the linear programming relaxation of the formulation. An empirical study demonstrates that the proposed approach can achieve better clustering accuracy than state-of-the-art methods when the data is high-rank or the percentage of missing data is high.

1. Introduction

Given a partially observed data matrix $X \in \mathbb{R}^{d\times n}$, where the column vectors of X are assumed to lie on or near a union of low-dimensional subspaces, *subspace clustering with missing data* (SCMD) is the task of identifying clusters of vectors belonging to the same subspace. Column vectors, $X_1, X_2, \dots X_n$ of X are sampled from a union of K subspaces, $\bigcup_{i=1}^{K} \mathcal{S}_i$, where each of the subspaces $\mathcal{S}_i$ is of dimension $r < d$. The SCMD problem has applications in many areas such as image classification [16, 29], motion segmentation [23, 24], and recommendation systems [22]. In this short paper, we describe a computationally-effective approach for SCMD that is based on integer programming.

Prior Work. In the last few years, many innovative methods for SCMD have been proposed [15]. Self-expressive methods, originally proposed for complete data by Elhamifar and Vidal [7], have been extended to the case of missing data [5, 8, 9, 17, 25, 27]. Self-expressive methods may have trouble correctly clustering the data points when the percentage of missing data is high or the matrix is high-rank, i.e., when $Kr \approx d$. Another family of methods is based on matrix factorization that directly seeks the bases of the low-dimensional subspaces [2, 20]. The resulting optimization problems are non-convex, and thus these methods are prone to converge to locally-optimal solutions. Lane et al.

[15] did an extensive empirical evaluation of existing SCMD algorithms and concluded that zero filled sparse subspace clustering methods (based on self expressiveness) [27], when alternated with low-rank matrix completion [17], a method called Alt-PZF-EnSC+gLRMC, showed the overall best performance. A disadvantage of Alt-PZF-EnSC+gLRMC is that it requires setting two regularization hyperparameters. Mixed integer linear programming (MILP)-based methods for subspace clustering have not been extensively explored. One exception is the work of Hu et al. [12], who give an integer programming model for subspace clustering. However, the approach does not account for missing data, assumes that candidate subspaces are explicitly enumerated as input to the model, and does not scale to large instances.

Paper contributions. We propose a novel MILP solution framework for the SCMD problem that is based on dynamically determining a set of candidate subspaces and optimally assigning data points to the closest selected subspace. A key challenge in this approach is identifying, in a rigorous manner, a suitable set of candidate subspaces to include in the formulation. We cast this subspace generation problem as a nonlinear nonconvex optimization problem and propose a gradient-based approximate solution approach. Our framework can readily accommodate a huge number of candidate subspaces through its use of Benders decomposition to solve the linear programming (LP) relaxation of the MILP. The model has the advantage of integrating the subspace generation and clustering in a single, unified optimization framework without requiring any hyperparameter tuning. Our computational results reveal that the proposed method can achieve higher clustering accuracy than state-of-the-art methods in certain data regimes.

2. Integer Programming Formulation

We assume that we are given a data matrix $X \in \mathbb{R}^{d\times n}$, with missing entries, whose columns are concentrated near a union of K subspaces, and each subspace is of dimension r. We let $\Omega \in \{0,1\}^{d\times n}$ be the indicator matrix of observed entries for X, and we denote the set of integers $\{1,2,\ldots,T\}$ as $[T]$. Our approach is based on iteratively building a collection of T candidate subspaces, where for each candidate subspace $t \in [T]$, we let $U_t \in \mathbb{R}^{d\times r}$ be a basis for its column subspace. Integer programming is then employed to simultaneously select the best set of K candidate subspaces and assign each column of X to its closest selected subspace.

We define the *closeness* c_{jt} of the vector $X_j, j \in [n]$ to a candidate subspace $t \in [T]$ as its residual (squared-distance) on the observed entries which has a closed form solution [2]:

$$c_{jt} := \min_{v\in\mathbb{R}^r} \left\{ \sum_{i:(i,j)\in\Omega} (X_{ij} - (U_t v)_i)^2 \right\} = \|(X_j)_\Omega - P_{(U_t)_{\Omega,j}}(X_j)_\Omega\|_2^2. \tag{1}$$

Here $U_{\Omega,j}$ denotes the restriction of the subspace with basis U to the rows observed in column j and $P_{U_{\Omega,j}}$ denotes the projection operator, $P_{U_{\Omega,j}} := U_{\Omega,j}(U_{\Omega,j}^T U_{\Omega,j})^{-1} U_{\Omega,j}^T$. Given T candidate subspaces, we formulate the SCMD problem as an integer program. Let $x_{jt} \in \{0,1\}, \forall j \in [n], t \in [T]$ be a binary assignment variable that determines if vector j is assigned to subspace t, and $z_t \in \{0,1\}, \forall t \in [T]$ be a binary selection variable that indicates whether subspace t is selected. Our complete integer programming formulation is given in (2). Objective (2a) ensures that the model looks for the least cost assignment of vectors and subspaces. Constraints (2b) ensures that each vector is assigned to exactly one subspace, and constraints (2c) enforce that a vector is assigned to only a selected subspace. Constraint (2d) ensures that exactly K subspaces are selected. The assignment of points to selected subspaces is similar to the facility location problem [26], where the goal is to select facilities to open and assign each customer to an open facility. In our SCMD formulation, subspaces play the role of facilities, and vectors play the role of customers.

$$\min_{x\in\{0,1\}^{n\times T}, z\in\{0,1\}^T} \sum_{t\in[T]}\sum_{j\in[n]} c_{jt}x_{jt} \tag{2a, MILP}$$

$$\sum_{t\in[T]} x_{jt} = 1, \quad \forall j \in [n] \tag{2b}$$

$$x_{jt} \le z_t, \qquad \forall j \in [n], t \in [T] \tag{2c}$$

$$\sum_{t\in[T]} z_t = K, \tag{2d}$$

3. Decomposition algorithm

The formulation (2) is solved via the well-known branch-and-bound method [14], which relies on solving a sequence of linear programming (LP) relaxations. The LP relaxation of (2) is the problem created by replacing the integrality conditions $z_t, x_{jt} \in \{0,1\}$ with simple bound constraints $z_t, x_{jt} \in [0,1]$. The optimal solution value of the LP relaxation provides a lower bound on the optimal solution to (2). The optimal dual variables of the LP relaxation also provide a systematic mechanism for dynamically generating new candidate subspaces—a vital component of our solution framework. Because the number of candidate subspaces T and the number of points n may be quite large, solving the LP relaxation is a computational challenge. In Section 3.1, we discuss a problem-specific implementation of the Benders decomposition method for the solution of the LP relaxation to (2). Section 3.2 describes how to dynamically generate improving candidate subspaces.

3.1. Row generation

Benders decomposition is a technique that enables solution of extremely large LP problems that have special structure [4]. It has been applied to large-scale facility locations by Fischetti et al. [10], and our formulation has the same structure. We eliminate x_{jt} variables (# $n \times T$) and add a set of continuous variables w_j (#n) representing the assignment cost for vector $j \in [n]$. The resulting reformulation of the LP relaxation of (2) is

$$\min_{w,z}\Big\{\sum_{j\in[n]} w_j \ : \sum_{t\in[T]} z_t = K, w_j \ge \Phi_j(z)\ \forall j \in [n], z_t \in [0,1]\ \forall t \in [T]\Big\}. \tag{3}$$

The function $\Phi_j(z)$ gives the minimum assignment cost for the vector $j \in [n]$ to a collection of subspaces parameterized by the variables $z \in [0,1]^T$. Note that the components of z may take fractional value. Specifically, $\Phi_j(z), j \in [n]$ is calculated by the following *subproblem*:

$$\Phi_j(\hat z) = \min_x\Big\{\sum_{t\in[T]} c_{jt}x_t : \sum_{t\in[T]} x_t = 1, 0 \le x_t \le \hat z_t, \forall t \in [T]\Big\}. \tag{4}$$

The function $\Phi_j(z)$ is piecewise-linear and convex, and Benders decomposition works by dynamically building a lower-bounding approximation to $\Phi_j(z)$. Moreover, its evaluation also gives sufficient information from which to create a lower-bounding approximation. The optimization problem (4) used to evaluate $\Phi(\cdot)$ has a closed-form solution. We refer reader to Appendix A.1 for more details.

3.2. Column generation

In our discussion to this point, we have assumed that we are given T candidate subspaces. However, in reality, there are infinitely-many subspaces to consider. Let $\mathcal{T}$ be the set of all subspaces. Key to our approach is a *column generation* method for dynamically identifying new subspaces that have the potential to improve the solution to (2). Column generation is a classical method for LP [11] that also has seen significant use in solving MILP problems [3]. The key idea behind column generation is to create an auxiliary problem, called the *pricing problem*, whose solution identifies if there is an additional variable (a candidate subspace), that, when added to the LP (3), could improve its solution value. The formulation of the pricing problem follows naturally from LP duality theory. If the *reduced cost* of a column (subspace variable) is negative, then, by increasing the value of that variable from its nominal value of zero, the objective value of the LP may improve. Thus, we should seek columns (subspaces) with negative reduced cost. If all columns have non-negative reduced cost, the current solution of the LP with the subset T of candidate subspaces is optimal to the true problem containing all subspaces $\mathcal{T}$.

Given the optimal dual variables (β, α) to the solution of (3) (details in Appendix A.1, (8)), the reduced cost of a column/subspace variable z_t is given by the formula

$$-\beta - \sum_{j \in [n]} \sum_{i \in [p_j]} \alpha_{ji} \max\{c^*_{ji} - c_{jt}, 0\}, \tag{5}$$

where c_{jt} is the assignment cost of column vector X_j onto subspace t. Recall (1), that describes the c_{jt} as a function of the basis matrix, $c_{jt} := h_j(U_t) := \min_{v \in \mathbb{R}^r} \left\{ \sum_{i:(i,j)\in\Omega} (X_{ij} - (U_t v)_i)^2 \right\}$. Thus, to obtain a column of minimum reduced cost, we can solve the following pricing problem to identify the subspace basis matrix:

$$\max_{U \in \mathbb{R}^{d \times r}} g(U) = \sum_{j \in [n]} \sum_{i \in [p_j]} \alpha_{ji} \max\{c^*_{ji} - h_j(U), 0\}. \tag{6}$$

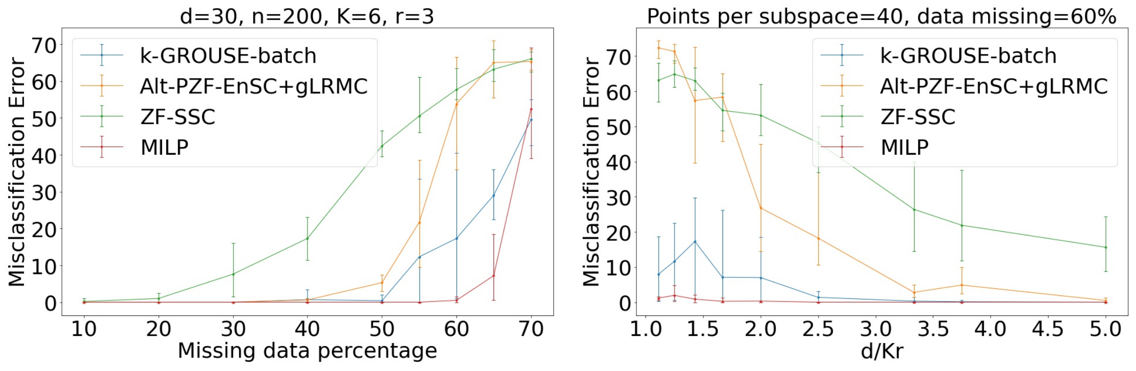
The optimization problem (6) is not convex, and hence is difficult to solve to provable global optimality. We find local solutions to (6) with a gradient-based approach discussed in Appendix A.2.

Row and column generation is performed iteratively until no new negative reduced cost columns are found. We refer reader to Appendix A.3 for details of our unified framework.

4. Computational study

4.1. Synthetic experiments

Experimental Setup. We construct K random subspaces with bases $U_k \in \mathbb{R}^{d\times r}, \forall k \in [K]$. Each entry of U_k is sampled from a standard Gaussian. Each data vector $j \in [n]$ is sampled from one of the K subspaces, i.e., $X_j = U_k v_j$ for a random $k \in [K]$ where $v_j \in \mathbb{R}^r$ is sampled from a standard Gaussian. We drop a certain fraction f of the entries of the data matrix X yielding the set of observed entries Ω. The choices of d, K, and r in our synthetic experiments are similar to the synthetic experiments in literature [15, 27]. We benchmark our MILP approach against zero filled sparse subspace clustering algorithm (ZF-SSC) [27], Alt-PZF-EnSC+gLRMC [15], and k-GROUSE [2]. The k-GROUSE algorithm is initialized with the output clusters from ZF-SSC, and Alt-PZF-EnSC+gLRMC is initialized with output clusters from ZF-EnSC [28]. All methods are tuned for best performance with parameters configurations reported in Appendix A.4. We evaluate the above algorithms using misclassification error which is defined as the ratio of the number of misclassified points to the total number of points. The best mapping between ground-truth and predicted clusters is found using Hungarian algorithm [13]. We used Gurobi 8.1 as the MILP solver with a time limit of 5 hours for each MILP run. These experiments were performed on a 4 core 16 GB machine.



(a) Effect of missing data

(b) Effect of d/Kr

Figure 1: Misclassification error as a function of missing data and d/Kr. Error bars plotted using minimum and maximum errors.

Effect of missing data fraction. We study the effect of missing data on misclassification error. We fix $d = 30, n = 200, K = 6, r = 3$, and vary f between 10-70%. Figure 1(a) shows the misclassification error as a function of the missing data fraction. We observe that ZF-SSC exhibits significantly increased misclassification error already for $f > 20\%$. Alt-PZF-EnSC+gLRMC and k-GROUSE perform similarly and have increased classification error when f is larger than 50-55%. The MILP approach on the other hand successfully classifies most instances up to f around 65%. In the high-missing data regime (40-70%), MILP yields the smallest misclassification error.

Effect of ambient dimension and total dimension of subspaces. We next study the effect of total dimension of the data ($K \times r$) relative to the ambient dimension d. We remove $f = 60\%$ of the data and we fix $n = 40K$. We choose d, r, K such that $d/(Kr) \in [1, 5]$. We show the misclassification error for each method with respect to $d/(Kr)$ in Figure 1(b). Since ZF-SSC does not perform well with high missing data ($f = 60\%$), we find that ZF-SSC gives high misclassification errors in all cases. Performance of the algorithms improve as we move towards the low-rank regime. In low-rank regime ($d/(Kr) \in [2.5, 5]$), both MILP and k-GROUSE give perfect classification and Alt-PZF-EnSC+gLRMC performs well. In the high-rank regime ($d/Kr \in [1, 2.5]$), we observe that only MILP gives near perfect classifications while Alt-PZF-EnSC+gLRMC has high misclassification. Computational times for these experiments are reported in Appendix A.5.

5. Conclusion

We proposed a novel MILP framework for the subspace clustering with missing data (SCMD) problem and showed its effectiveness relative to other state-of-the-art methods, especially in certain instance regimes. Our MILP framework gives the user flexibility to use a different function for cost of assignment between vector and subspace and can also easily be extended to include constraints. A direction for future work is to investigate techniques for dynamically determining the dimensions of the subspaces used in the formulation. One simple idea in this direction is a sequential framework, where one starts with dimension one subspaces and consecutively moves towards high dimensions while removing points which can be explained with low dimensional subspaces.

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Appendix A.

A.1. Benders decomposition

Recall from Section 2 that c_{jt} represents *closeness* of the vector $X_j, j \in [n]$ to a candidate subspace $t \in [T]$. Let $\{\sigma_1^j, \dots, \sigma_T^j\}$ be a permutation of $\{1, \dots, T\}$ satisfying $c_{j\sigma_1^j} \leq c_{j\sigma_2^j} \leq \dots \leq c_{j\sigma_T^j}$, and let $t_j^* := \min\{t : \sum_{s=1}^t \hat{z}_{\sigma_s} \geq 1\}$ be the *critical index*. As described in [10], the *Benders cut* that can be used to lower-approximate the function $\Phi_j(\cdot)$ is

$$w_j + \sum_{i=1}^{t_j^*-1} (c_{j\sigma_{t_j^*}^j} - c_{j\sigma_i^j}) z_{\sigma_i} \geq c_{j\sigma_{t_j^*}^j}. \tag{7}$$

These inequalities are accumulated iteratively. Let p_j denote the number of Benders cuts included in the model at the current stage in the algorithm for each $j \in [n]$. Let t_{ji}^* denote the critical index for vector $j \in [n]$ associated with Benders cut $i \in [p_j]$, and let $c_{ji}^* := c_{j\sigma_{t_{ji}^*}^j}$ denote the critical cost for the j^{th} vector in cut $i \in [p_j]$. The Benders *master problem* is then

$$\min_{w,z} \sum_{j \in [n]} w_j \tag{8}$$

$$\sum_{t \in [T]} z_t = K, \tag{β}$$

$$w_j + \sum_{\ell=1}^{t_{ji}^*-1} (c_{ji}^* - c_{j\sigma_\ell^j}) z_{\sigma_\ell^j} \geq c_{ji}^*, \forall j \in [n], i \in [p_j], \tag{α_{ji}}$$

$$0 \leq z_t \leq 1, \qquad \forall t \in [T]. \tag{μ_t}$$

Here β, α, and μ are dual variables corresponding to the respective constraints, and play an important role in the column generation process described in Section 3.2. Solving (8) gives a solution $(\hat{w}, \hat{z})$. The subproblem (4) is then solved to evaluate $\Phi_j(\hat{z})$ for each $j \in [n]$ and to generate new Benders cuts (7). If $\Phi_j(\hat{z}) = \hat{w}_j$, then the generated inequality does not improve the approximation to $\Phi_j(\cdot)$, and the cut is not added to (8). The Benders procedure stops when no new cuts are added. At this point, the LP relaxation of (2) is solved.

A.2. Pricing problem

Gradient-based approach for pricing problem. Recall from Section 3.2,

$$c_{jt} := h_j(U_t) := \min_{v \in \mathbb{R}^r} \Big\{ \sum_{i:(i,j) \in \Omega} (X_{ij} - (U_t v)_i)^2 \Big\}.$$

If $h_j(U) \neq c_{ji}^* \; \forall j \in [n], i \in [p_j]$, then the function $g(U) = \sum_{j \in [n]} \sum_{i \in [p_j]} \alpha_{ji} \max\{c_{ji}^* - h_j(U), 0\}$, described in (6) is differentiable. For notational convenience, we use the indicator parameter $\hat{y}_{ji} = 1$ if $c_{ji}^* > h_j(\hat{U})$ and 0 otherwise. The partial derivative of $g(\cdot)$ with respect to matrix element U_{ab} evaluated at $\hat{U}$ is given by

$$\frac{\partial g(\hat{U})}{\partial U_{ab}} = \sum_{j \in [n]} \sum_{i \in [p_j]} 2\hat{y}_{ji}\alpha_{ji} \sum_{\ell \in \Omega_j} (X_{\ell j} - \hat{u}_\ell^\top \hat{v}_j)\hat{v}_{jb} \quad \forall a \in [d], b \in [r]. \tag{9}$$

Here $\hat{u}_\ell$ represents ℓ^{th} row of basis $\hat{U}$, and $\hat{v}_j$ is the minimizer in (1) for $U_t = \hat{U}$. If $h_j(\hat{U}) = c^*_{ji}$ for some $j \in [n], i \in [p_j]$, we can still apply the formula (9) to obtain an element of the generalized subdifferential for $g(\cdot)$ at $\hat{U}$ [6].

Algorithm 1: Gradient-based approach for locally solving pricing problem

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Data: $X_\Omega$, subspaces dimension ($r$), critical costs $c^*_{ji}$ and dual solution $\alpha_{ji}, \forall j \in [n], i \in [p_j]$
Input: $U^0$ ;                                        /* initial subspace */
Output: Subspaces generated from each iteration
1  $\hat{U} \leftarrow U^0$ ;
2  while not converged
3      for $j = 1, 2, \ldots, n$ do
4          for $i = 1, 2, \ldots, p_j$ do
5              $\hat{y}_{ji} \leftarrow 1$ if $c^*_{ji} > \|(X_j)_\Omega - P_{\hat{U}_{\Omega,j}}(X_j)_\Omega\|_2^2$, 0 otherwise ;
6          end
7      end
8      Calculate $\nabla g(\hat{U})$ using (9) ;
9      $\hat{g} \leftarrow \sum_{j\in[n]} \sum_{i\in[p_j]} \alpha_{ji}\hat{y}_{ji}c^*_{ji}$, $\hat{\gamma} \leftarrow \frac{\hat{g}-g(\hat{U})}{\|\nabla g(\hat{U})\|_2^2}$ ;     /* Polyak step size */
10     $\hat{U} \leftarrow \hat{U} + \hat{\gamma}\nabla g(\hat{U})$ ;        /* move in positive gradient direction */
11 end
```

We outline our gradient-based approach in Algorithm 1. In each iteration, we calculate $\hat{y}$ (lines 4-6), the gradient (line 8), and move in the positive gradient direction (line 10). We use the Polyak step size [21], and this requires an estimate of the optimal value of objective function. We approximate the optimal value g^*, as $g^* \approx \sum_{j\in[n]} \sum_{i\in[p_j]} \alpha_{ji}\hat{y}_{ji}c^*_{ji}$ (line 9). Empirical experiments show that this choice of step size works well. In our implementation, we terminate the algorithm (line 2) after a maximum of 2000 iterations, if $\|\nabla g(\hat{U})\|_2 < 0.0001$, or if $g(\hat{U})$ has not improved by at least $\epsilon = 0.01$ in the last 100 iterations. Of all the columns generated, only those with negative reduced cost as calculated in (5) are added to the master problem (8).

A.3. Unified Framework

We discuss a unified MILP framework for SCMD problem that integrates the use of Benders decomposition and column generation. We also point out that we generate new columns (z_t variables) only at the initial LP relaxation (the so-called root node), and not at additional nodes in the branch-and-bound tree. We describe the overall MILP framework in Algorithm 2. We initialize the algorithm with m randomly generated subspaces (line 1) to initialize model (8). We then solve the master LP relaxation (8) in line 7, and generate Benders cuts for each $j \in [n]$ (lines 8-11). We repeat this until no violated cuts are found (line 10). We then proceed to generate new columns by solving the pricing problem (6). Because we use a gradient-based approach to solve the nonconvex problem (6), we initialize Algorithm 1 with different random choices of U^0 (lines 14-18) to identify different locally-optimal solutions. Each U^0 is obtained by selecting a random subset of $2r$ vectors from the $3Kr$ vectors that have the largest $\hat{w}_j$ values in the current LP solution of the master problem (8). Then, we use a fast low-rank matrix completion algorithm [1] to find the basis U^0 for a best-fit subspace for these vectors. We repeat this until we have tried at least K different U^0 and have found at least one negative reduced cost column, or reach the maximum allowed iterations. If negative reduced cost columns are found, we add them to the master LP, delete existing Benders cuts since they become invalid due to new z_t variables (lines 21-22), and return to the process of generating Benders cuts. We repeat this process as long as we are able to generate new columns.

Once we fail to find a negative reduced cost column, we exit the root node loop and pass the updated MILP model with the new columns and cuts included to a MILP solver (we use Gurobi [19]). We also provide a callback routine to the solver to generate Benders cuts as necessary when an integer solution is found during the branch-and-bound tree search. We use the solution from the MILP solver to determine the selected subspaces and assignments of vectors to them (line 27).

A.4. Parameters choice for state-of-the-art methods

We tuned the methods against which we compare our MILP approach for different parameter configurations. We report these configurations in Table 1. The range for each parameter is based on the recommendations from the original papers [2, 15, 27]. The best parameter configuration is selected based on the least misclassification error.

A.5. Computation time

We report average computation times for some of the instances we considered in Section 4.1 in Table 2 . As expected, ZF-SSC and k-GROUSE are significantly faster than the other two methods, with computing times ranging from 2-6 minutes, whereas the computing time ranges between 45 minutes and 3.1 hours for the MILP approach and between 11 minutes and 11.4 hours for Alt-PZF-EnSC+gLRMC. For Alt-PZF-EnSC+gLRMC, the time reported includes the necessary time for hyperparameter tuning and hence is much higher than ZF-SSC since it involves solving LRMC and EnSC repeatedly for different set of parameters. We give a detailed breakdown of the computation time spent in each component of the MILP approach in Table 3. A large portion of the computation in the MILP approach ($\approx 50\%$) is spent computing the c_{jt} coefficients and about 20% is spent calculating gradients, which are both operations that can be done in highly parallel fashion. Thus, while we did not pursue a parallel implementation, we expect significant speedups are possible. Overall, these results indicate that the MILP framework is feasible for moderate size data and competitive with Alt-PZF-EnSC+gLRMC, but is best suited for applications where one desires accuracy over speed, e.g., predicting gene-disease association [18].

We report detailed computational times of the MILP approach in Table 3. For the same instances reported in Table 2, we report the number of new columns generated from solving the pricing problem (# New cols), the number of Benders cuts at the Root node and in the B&B (Callback), the number of nodes explored in B&B tree, the MILP total solution time (Total sol. time), the time spent solving the pricing problem (Pricing), the time spent solving the Benders LP (Benders LP), the time spent in B&B tree (B&B), and the time spent in calculating residuals costs (Residuals). All numbers reported in Table 3 are averaged over five different random instances.

We point out that the majority of the computational time in our MILP approach is spent in calculating residuals costs and solving the pricing problem many times. The time spent on processing the B&B tree is very small ($\approx 1\%$) of the total MILP time when no Benders cuts are generated in B&B tree with the callback. When Benders cuts are required to be generated in the B&B tree, then the time spent processing the B&B tree goes up as well. We also observe that in most cases

Algorithm 2: Unified MILP framework for SCMD

```
Input: $X_\Omega$, subspaces dimension $r$, number of subspaces $K$
Output: Segmentation of columns of $X$ in $K$ clusters: $S_k$ and basis $U_k, \forall k \in [K]$
1  Generate $m = 500 * K$ random subspaces to initialize MILP model (3) ;
2  root node continue $\leftarrow True$, generate cuts $\leftarrow True$;
3  while root node continue
4      root node continue $\leftarrow False$ ; /* switched back on if new columns found */
       // generate Benders cuts
5      while generate cuts
6          generate cuts $\leftarrow False$ ;
7          solve master LP relaxation (8) to obtain $\hat{z}$;
8          for $j = 1, 2, \dots, n$ do
9              Generate and add Benders cuts of the form (7) to master (8);
10             if cuts found: generate cuts $\leftarrow True$ ;
11         end
12     end
       // generate new columns
13     $T_n \leftarrow \emptyset$;
14     for $it = 1, \dots, maxIt = 50$ do
15         $U^0 \leftarrow$ best fit subspace on randomly sampled $2 * r$ vectors from costliest $3 * K * r$ vectors;
16         Solve pricing problem using Algorithm 1 and add columns with negative reduced cost to $T_n$;
17         if $|T_n| \geq 1$ and it $\geq K$: break;
18     end
19     if $T_n \neq \emptyset$
20         Calculate residual cost $c_{jt}, \forall j \in [n], t \in T_n$ using (??);
21         $[T] \leftarrow [T] \cup T_n$ ;                    /* Add new $z_t, t \in T_n$ variables */
22         Remove all Benders cuts from (8) ;      /* invalid due to new $z_t$ vars */
23         root node continue $\leftarrow True$;
24     end
25 end
26 $\hat{x}, \hat{z} \leftarrow$ Solve MILP model (3) with Gurobi, give a callback routine for Benders cuts;
27 return $\{S_t = \{j \in [n] : \hat{x}_{jt} = 1\}, U_t, \forall t \in [T]$s.t. $\hat{z}_t = 1\}$ ;
```

Table 1: Parameters choice for evaluated algorithms

Algorithm	**Parameters**
ZF-SSC [27]	$\lambda \in \{10^{-3}, 10^{-2}, 0.1, 1, 10, 10^2, 10^3\}$
Alt-PZF-EnSC+gLRMC [15]	$\lambda \in \{5, 50, 300\}, \gamma \in \{0.5, 0.7, 0.9\}$
k-GROUSE [2]	$\eta_0 = 0.1$, diminishing step size

Table 2: Comparison of computation time (s). Instance notation: I-30-200-6-3-40 stands for $d = 30, n = 200, K = 6, r = 3, f = 40\%$

Instance	**ZF-SSC**	**Alt-PZF-EnSC+gLRMC**	**k-GROUSE**	**MILP**
I-40-200-4-3-60	118.6	1591.2	153.6	2680.4
I-30-200-6-3-20	163.7	672.3	181.9	3830.3
I-30-200-6-3-40	155.6	791.2	190.7	4127.3
I-30-300-6-3-65	338.9	29205.9	347.2	12299.4
I-30-360-9-3-60	290.2	41123.5	291.2	11117.7

Table 3: Detailed computational times for MILP

Instance	# New cols	# Benders cuts		# B&B nodes	Time (s)				
		Root node	Callback		Total sol. time	Pricing	Benders LP	B&B	Residuals
I-30-120-6-3-65	140241	11663	2558	11	19739.4	7944.2	126.1	755.5	6977.9
I-30-160-4-2-60	37104	7167	0	0	2002.5	998.7	7.6	2.2	997.9
I-30-160-4-3-60	49982	7071	0	0	3043.6	1426.3	9.3	3.4	1382.0
I-30-180-6-3-65	146253	14334	144	0	18751.7	8188.4	143.1	146.0	7313.3
I-30-200-5-3-60	51252	8156	0	0	2841.3	1410.3	9.2	3.2	1419.1
I-30-200-6-3-10	28246	8135	0	0	4177.5	1550.6	16.0	3.3	1736.3
I-30-200-6-3-20	33257	8998	0	0	3816.7	1440.7	30.2	2.7	1625.3
I-30-200-6-3-30	30042	10090	0	0	3417.2	1286.6	15.6	2.3	1410.0
I-30-200-6-3-40	34774	9163	0	0	4111.3	1741.6	16.4	3.9	1892.0
I-30-200-6-3-50	37211	8316	0	0	4125.6	1820.0	16.9	4.9	1937.3
I-30-200-6-3-55	50338	9613	0	0	5799.6	2663.3	19.1	4.0	2727.6
I-30-200-6-3-60	64502	10073	0	0	7058.0	3486.1	32.6	7.6	3458.6
I-30-200-6-3-65	118255	12552	120	0	14453.6	6834.9	94.9	164.8	6420.3
I-30-200-6-3-70	135352	11627	2560	11	21699.9	9593.6	129.1	2010.8	8702.6
I-30-240-6-3-60	57925	11514	0	0	4125.6	2010.9	16.4	4.1	2012.4
I-30-240-6-3-65	87303	12461	0	0	11813.3	5821.1	61.5	15.9	5598.9
I-30-280-7-3-60	71170	13946	0	0	4538.0	2169.4	22.5	7.4	2059.2
I-30-300-6-3-65	67051	15379	0	0	12272.0	5872.8	69.1	18.5	5763.1
I-30-320-8-3-60	79744	16073	188	0	15189.5	7779.2	166.9	154.7	7676.5
I-30-360-6-3-65	60075	15951	0	0	6924.0	3620.2	29.5	8.0	3603.9
I-30-360-9-3-60	86972	19349	0	0	11082.7	5647.4	72.8	15.3	5488.2
I-30-60-6-3-65	54863	5026	432	8	4114.2	1490.8	32.2	88.3	1385.7
I-40-160-4-2-60	28746	6071	28	0	4250.8	2077.7	14.6	3.2	2158.3
I-40-200-4-3-60	31498	7561	0	0	2673.6	1217.2	8.0	2.5	1286.2

the problem is solved without branching at all—the number of B&B nodes is 0. The time spent in solving LP relaxations while generating Benders cuts is also very small.