The moment-LP and moment-SOS approaches in polynomial optimization

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We discuss the optimization problem $\mathbf{P} : \inf \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \}$ where f is a polynomial and $\mathbf{K} \subset \mathbb{R}^n$ is the basic closed semi-algebraic set (assumed to be compact)

$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^n : \quad g_j(\mathbf{x}) \geq 0, \quad j = 1, \dots, m \}$$

for some polynomial g_j, $j = 1, \dots, m$. When one focuses on the *global* minimum f^* (as opposed to a local optimum), problem $\mathbf{P}$ can be written as:

$$f^* = \sup \{ \lambda : \quad f(\mathbf{x}) - \lambda \geq 0, \quad \forall \mathbf{x} \in \mathbf{K} \}.$$

When f is a polynomial and $\mathbf{K}$ is a compact basic semi-algebraic set, powerful positivity certificates of Real Algebraic Geometry allow to express the difficult positivity constraint "$f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbf{K}$" in a way that can be exploited for efficient numerical computation. Indeed one may then define a hierarchy of *convex relaxations* $(\mathbf{P}_k)$, $k \in \mathbb{N}$, of $\mathbf{P}$ which provides a monotone sequence of upper bounds ($f_k^* \geq f^*$) such that $f_k^* \to f^*$ as $k \to \infty$. Depending on which type of positivity certificate (e.g. one due to Krivine, Handelman and Vasilescu or one due to Schmüdgen and Putinar) one ends up with solving a hierarchy of either *LP-relaxations* or *SDP-relaxations* (or semidefinite relaxations). In both cases the resulting convex relaxation $\mathbf{P}_k$ becomes more and more difficult to solve as its size increases with k.

We then discuss the relative merits and drawbacks of both hierarchies of LP- and SDP-relaxations and their impact not only in optimization (and particularly combinatorial optimization) but also in many areas for solving instances of the so-called *Genaralized Problem of Moments* (GMP) with polynomial data (of which Global Polynomial Optimization is in fact the simplest instance).

Finally, we also introduce another characterization of nonnegativity on a closed set $\mathbf{K} \subset \mathbb{R}^n$ which can be also exploited to now define a monotone non increasing sequence of upper bounds (f_k^*), $k \in \mathbb{N}$, that converges to the global minimum f^* of $\mathbf{P}$ as $k \to \infty$. Computing each upper bound f_k^* now boils down to solving a generalized eigenvalue problem associated with some pair of real symmetric matrices whose size increases with k.

References

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